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LETTER TO THE EDITOR

Hierarchical lattice with competing interactions: an example of a nonlinear map

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Abstract. An Ising model with competing nearest and more-than-nearest neighbour interactions on a hierarchical lattice is solved by decimation. In the context of nonlinear mappings the renormalisation group trajectories are analysed as the competition parameter is varied. Commensurate and incommensurate phases are identified.

Models with competing interactions have attracted a great deal of attention recently in the context of spatially modulated structures and/or spin-glass phenomena. For example, the ANNNI (axial next-nearest neighbour Ising) model has been extensively studied by a large variety of methods, such as the exact low-temperature expansion (Fisher and Selke 1980), Monte Carlo simulations (Selke and Fisher 1979, Selke 1981), mean field theories (Bak and von Boehm 1980), free fermion approximation (Villain and Bak 1981) and renormalisation group approaches (Švrakić 1982). The results of these studies show that the presence of competition gives rise to a remarkable richness of phenomena such as the spatially modulated ordering, possibly a devil's staircase like behaviour of the wavevector of modulation, a sequence of infinitely many distinct commensurate phases springing from a multiphase point, a Lifshitz point. In the ANNNI model the competition arises from nearest neighbour ferromagnetic and axial next-nearest neighbour antiferromagnetic interactions, each interaction preferring a different periodicity of the ordered phase. A different mechanism of competition is offered by spin-glass models where one has randomly distributed ferromagnetic and antiferromagnetic interactions. For these models the concept of frustration is central in understanding the unusual behaviour of the spin-glass phase.

In this article we consider the first type of competition. We study an Ising model with competing nearest and more-than-nearest neighbour interactions on a hierarchical lattice (Kaufman and Griffiths 1981) using the position space renormalisation group which is for this class of lattices exact. In the context of nonlinear mappings (Collet and Eckmann 1980, Ott 1981) the renormalisation group trajectories are analysed as the competition parameter is varied. A similar study of the spin-glass type of competition was recently performed by McKay *et al* (1982) and Erzan (1982).

The motivation for this study is twofold. (i) Our non-trivial model with competing interactions can be easily solved exactly, in contrast to the ANNNI model on hypercubic

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lattices where only a part of the phase diagram for $d > 2$ is known exactly. (ii) The method of treatment and the results are intriguing in their own right since they involve the analysis of nonlinear recursion relations, a subject which is relevant in many domains of physics. As will be seen below, the map we use contains a control parameter which enters the recursion relation in a non-multiplicative way and it violates the Schwarzian condition, in contrast with maps for which the original scenario of Feigenbaum (1978) was developed.

The hierarchical models (Forgacs and Zawadowski 1982, Kaufman and Griffiths 1981, Berker and Ostlund 1979) form a class of exactly solvable lattice models which exhibit phase transitions at finite temperatures with non-classical exponents. In figure 1 we show an example for construction of one hierarchical model. Four primitive bonds are assembled in the unit shown by the arrow (see figure 1(a)), which is then further assembled into a self-similar unit, and so on. This process is repeated *ad infinitum*. The result is a self-similar, non-homogeneous lattice with infinite number of spins. Note that different spins will have different coordination number and that dimensionality of the lattice is difficult to define (Kaufman and Griffiths 1981).

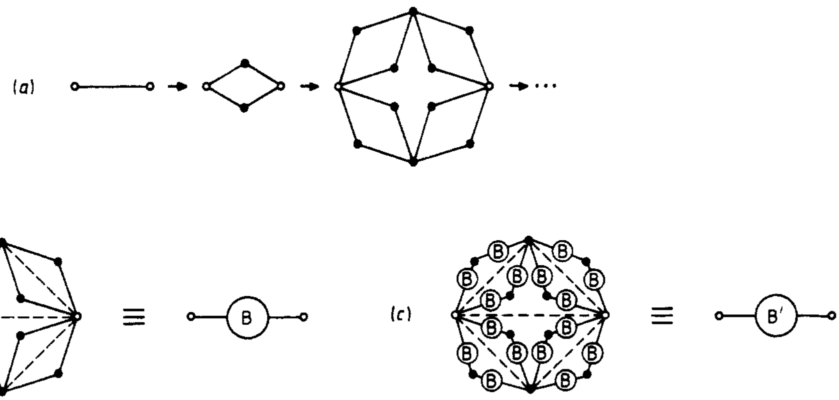

Figure 1. The construction of hierarchical models.

The construction can, of course, be reversed. A primitive bond is seen as a bond in a larger unit, which is itself a part in a larger self-similar unit, etc. The specific hierarchical model we use is constructed in this way and the process is illustrated in figures 1(b) and 1(c). The basic unit is shown in figure 1(b). The full lines are ferromagnetic couplings K, while the broken lines are the antiferromagnetic couplings $-pK$. In this way the competition between ferromagnetic and antiferromagnetic interactions is introduced. The parameter p measures the strength of competition and we shall always take it to be positive.

The construction proceeds as follows. The basic unit, denoted by B in figure 1(b), is assembled into the unit B′ shown in figure 1(c). The broken lines in this new structure are now $-pK'$, where K' is the effective interaction between two end (white) sites in figure 1(b), and is obtained by decimating over the black sites. In this way we can introduce the same amount of competition at each level of hierarchy. This procedure is then repeated infinitely many times. Obviously, the hierarchical lattices are highly artificial, defined, in essence, by their solution. This is the price we have to pay in order to obtain the exact solution.

The recursion relation for the coupling constant is readily obtained by decimation over black sites. We get

$$K_{n+1} = -pK_n + \ln\cosh 2(-pK_n + \ln\cosh 2K_n), \tag{1}$$

where K_n is the effective interaction between two nearest neighbour spins on the nth level of hierarchy. In a similarly straightforward manner one can calculate the constant-term recursion relation from which the free energy can be obtained. In the following we shall study the basic equation (1). Clearly, it is a one-dimensional nonlinear map with the parameter of competition p as the control parameter. Denoting the right-hand side of the equation (1) by a function f, we have the mapping $K_{n+1} = f(K_n, p)$. In order to analyse this map we note that it cannot be restricted to a finite interval for $p = 1$. We thus consider separately the cases $p < 1$ and $p > 1$.

For $p < 1$, f has one extremum (minimum) of the quadratic type and two fixed points: the trivial one ($K_1^* = 0$), which is stable, and another (K_2^*) unstable one. There is an interval in the (K, p) plane in which f violates the Schwarzian condition (Collet and Eckmann 1980), indicating that there may exist other stable orbits with restricted basins of attraction. Indeed, we have found that for certain values of p and K there are stable limit cycles *in addition* to the above fixed point.

For $1 < p < \frac{4}{3}$, f is again a map on a finite interval (in contrast to $p = 1$). $K_1^* = 0$ becomes unstable for $p > 1$. At $p = \frac{4}{3}$ the minimum of f disappears. For $(17/9)^{1/2} < p < 2.56\ldots$ the map f is no longer finite and all points from this region flow to a $\pm\infty$ limit cycle. Beyond $p = 2.56\ldots$ a new sequence of limit cycles appears (f develops new extrema) up to $p = 5$ where the $\pm\infty$ limit cycle sets in again and remains for all higher values of p. More detailed properties of the mapping f are given below.

We now present numerical details of the mapping f and the main body of our results. Figure 2 depicts the global features of the flows. The physical interpretation is as follows: clearly $K_1^* = 0$ is always the fixed point of f with $f'(K_1^*) = -p$, where a prime denotes the first derivative. This fixed point corresponds to the paramagnetic phase. In the region $p < 1$ there is another fixed point (an unstable one) K_2^* given implicitly by the equation $p = (\ln\cosh 2K_2^*)/K_2^* - 1$. This condition determines the ferromagnetic phase boundary since the points $K > K_2^*$ flow to infinity, which is the sink of the ferromagnetic phase. Points $K < K_2^*$, for $p < p_1$ ($=0.6698\ldots$), flow to $K_1^* = 0$.

At $p = p_1$ a stable two-point limit cycle appears. The basin of attraction of this limit cycle has a band-like structure. To see this we fix the value of p at $p_1 < p < p_2$ (see figure 3). Now we start increasing the initial value of the coupling K, starting at zero. For sufficiently small K, K_n will flow to zero. This will happen until K reaches the value K_{L} above which it enters the range of attraction of the stable limit cycle. As the value of K is further increased, it reaches the value K_{U} above which it leaves the basin of attraction of the stable limit cycle and maps again onto zero. Upon further increase, above K_{U}, K enters the domain of attraction of the limit cycle for the second time at $K_{\mathrm{L}1}$ and leaves this domain at $K_{\mathrm{U}1}$. In such a sequence the domain of attraction of the limit cycle is entered until the value K_2^* is reached. This band structure is shown in figure 3. As K is varied between zero and K_2^* infinitely many bands are encountered. One may interpret the stable limit cycles as sinks of modulated phases commensurate with the hierarchical structure of the model, because the effective couplings repeat themselves after the period of the cycle. Then the band structure corresponds to infinitely many re-entrances separated by paramagnetic regions. A similar result for the three-state chiral Potts model on a hierarchical lattice was recently

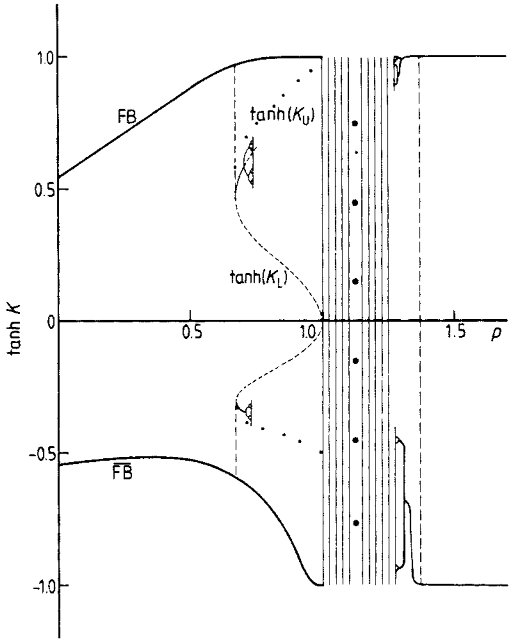

Figure 2. Renormalisation group topology as competition parameter p is varied showing some of the bifurcation cascades, chaotic bands (vertical segments) and windows (indicated schematically by large dots). Note that the line $\overline{\text{FB}}$ maps onto FB (ferromagnetic phase boundary) and points below $\overline{\text{FB}}$ map onto points above FB. For details in the range $0.67 < p < 1$ see figure 3.

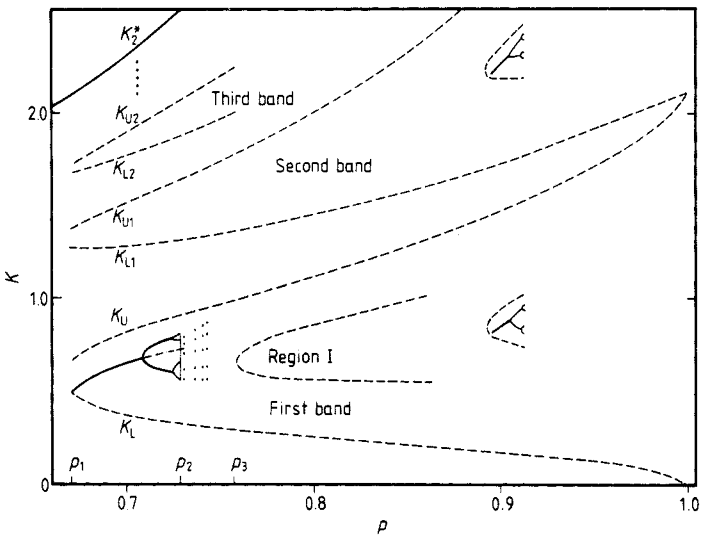

Figure 3. Details of the topology and basins of attraction (broken lines) for $0.67 < p < 1$ and $K > 0$. The dots illustrate windows in between chaos. Note that higher bands of the basin of attraction are not shown. The different basin of attraction near the 3×2^n cycles ($p \cong 0.9$) is shown schematically.

obtained by Huse (1981; figure 10). Note that K_L is the lower limit of the 'modulated phases'. K_L itself is an element of the two-point *unstable* limit cycle. In figure 3 we show only three bands. The points from within the upper bands map onto lower ones until the lowest band is reached. The points from this band (first band in figure 3) after a single iteration map to the negative side of the K axis and then back again into the first band. If one starts with $K<0$ the first iteration will map it onto the positive side and the above discussion applies unchanged. For this reason we show in figure 3 the band structure only for the positive values of K.

For $p>p_1$ one observes a cascade of bifurcations until the accumulation point at $p_2=0.7289\ldots$ is reached. The convergence rate δ (Collet and Eckmann 1980) after a few period doublings seems to converge to the universal number 4.669 .... Beyond p_2 the chaotic regime is encountered with usual noisy bands (Collet and Eckmann 1980, Ott 1981) and windows of stable limit cycles of various periodicities. In the same spirit as the stable limit cycles are interpreted as commensurate structures we can interpret the noisy bands as incommensurate structures. One might push the analogy even further: the intermittent behaviour preceding the onset of a stable limit cycle window (Ott 1981), where long regular sequences are separated by irregular bursts, may be interpreted as the phase where 'almost commensurate' regions are separated by 'discommensurations' or 'domain walls'. There are many (presumably infinitely many) windows. For example, we find at $p\cong 0.7298$ a stable limit cycle sequence with periods 36–72 in the window of width 7×10^{-7}; a 30–60–120 sequence at $p\cong 0.736$; an 8–16–32–64 sequence at $p\cong 0.7498$ and many more.

For $p_3<p<1$, we find a 3×2^n bifurcation cascade followed by a chaotic regime (Collet and Eckmann 1980) for $0.895<p<0.9125$, while elsewhere almost every initial coupling K, $K<K_2^*$, appears to be mapped onto zero (since we changed the p-values in discrete steps of about 10^{-4} we certainly cannot exclude additional stable orbits on a smaller scale). The band structure becomes increasingly complex by taking the larger values of p. To give an example, we discuss briefly the situation as p just exceeds p_3: there, the boundaries of the old bands (K_{Ln}, K_{Un}) still give rise to the unstable cycles, as before. Also, the points inside the upper bands still map onto the lower ones until one reaches the lowest or first band (K_L, K_U). However, there is a new sub-band structure inside the first band: points in region I (see figure 3) will be mapped onto $K_U<K<K_{L1}$ after two iterations, from where they are mapped finally to zero. In addition, inside the first band there are regions II, III, ..., which map after two iterations onto regions I, II, ... (not shown in figure 3).

At $p=1$ we encounter a special situation: the fixed point at $K_1^*=0$ becomes marginal, and the unstable fixed point K_2^* moves to infinity: the ferromagnetic phase disappears. In addition the trajectories become chaotic.

For $p>1$ the chaotic trajectories are interrupted by windows of stable limit cycles. The domain of attraction of these limit cycles becomes the whole K axis except for a set of zero measure. For example, for $p\cong 1.13$ a twelve-point limit cycle is found which merges into a six-point cycle at $p\cong 1.15$ and this one becomes then a three-point cycle at $p\cong 1.18$ (note the order of periodicities as p is increased). There are many more similar windows in this region. At $p=1.285\ldots$ an accumulation point of the stable 2^n-point limit cycle is found. With increasing p this merges into a two-point limit cycle at $p\cong 1.29$. This is shown in figure 2. This two-point limit cycle spreads out until $p=(17/9)^{1/2}$ where it becomes a stable two-point limit cycle between $+\infty$ and $-\infty$. This cycle may be interpreted as the sink of a new ordered phase reminiscent of the $\langle 2\rangle$ phase in the ANNNI model (Fisher and Selke 1980).

In this letter we defined an exactly solvable model with competing interactions on a hierarchical lattice. We introduced competing interactions both locally and of long-range type. The strength of the nearest to next-nearest neighbour coupling is given by the parameter p. Hierarchical lattices have been introduced to get exact realisations of the renormalisation group transformations which are approximate for Bravais lattices (Forgacs and Zawadowski 1982, Berker and Ostlund 1979, Yeomans and Fisher 1981). However, they can also be constructed without referring to common lattices to elucidate interesting physical concepts (McKay *et al* 1982, Erzan 1982)—similar to other uncommon lattices like the Cayley tree (Vannimenus 1981).

In our case we can study the occurrence of modulated phases of commensurate and incommensurate type due to the competing interactions. The basic renormalisation group equation (1) describes the effective coupling constant on successive levels of hierarchy. The modulation shows up in the sequence of coupling constants one encounters by iterating the map *ad infinitum*. Stable limit cycles correspond to commensurate phases and the noisy bands may be interpreted as sinks of incommensurate phases. Several interesting phenomena occur, such as continuously varying critical exponents and a sequence of infinitely many re-entrances of modulated structures separated by paramagnetic phases reminiscent of the similar behaviour found in the mock ANNNI model (Huse *et al* 1981) or the chiral Potts model (Huse 1981). The latter phenomenon stems from some mathematical subtleties associated with the one-dimensional map, equation (1), which do not occur in many of the well known maps like the logistic one (non-multiplicative control parameter, breakdown of Schwarzian condition). We therefore conclude that hierarchical models may not only add to the understanding of complex physical situations, but may also exhibit a lot of interesting mathematics.

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